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Weak Anisotropic Reflections

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INTRODUCTION

As discussed elsewhere in this volume, the angular dependence of the P-wave reflection coefficient $R_p(\theta)$ between planar isotropic media involves the S-wave properties of both media. This situation has raised the hopes of the petroleum industry that S-wave information about subsurface formations may be found cheaply, through the range dependence of P-wave reflection amplitudes.

However, the subsurface formations are invariably anisotropic, and it is intuitively obvious that the angular dependence of elastic wave velocities should modify the angular dependence of the quasi-P-wave reflection coefficient. This intuition is verified in numerical studies reported in Wright (1984, 1987), who demonstrated that the effect is nontrivial and may substantially interfere with the deduction of S-wave properties. Banik (1987) verified Wright's results analytically, with a closed analytical expression for the anisotropic reflection coefficient, in the limit of weak anisotropy and small angles. The present work generalizes these analytic results to larger angles, and discusses the extension of understanding beyond the numerical examples.

A very brief account of weak isotropic reflections is followed by a very brief account of weak anisotropic body wave propagation. The application to P-wave reflections between weakly anisotropic media follows, with derivations given in an Appendix.

Weak Elastic Reflections

The exact solution for the oblique P-wave reflection at a planar interface between two isotropic media is well-known; a good account is presented in Aki and Richards (1980, chapter 5). However, the exact result is so complex algebraically that it is difficult to grasp intuitively the physics contained in it. Furthermore, it depends upon compressional velocity (V_p), shear velocity (V_s), and density (ρ) in both media, which means that a numerical study necessarily involves the exploration of a six-parameter model space. Not even the choice of parameters is obvious, and inappropriate choices [e.g., Poisson's ratio, cf. Koefoed (1955)] have received much discussion.

Fortunately, in most contexts in exploration seismology, an appropriate simplification is available. Since at most reflecting interfaces, the contrast in elastic properties is small, it is appropriate to linearize the exact solution in the small quantities:

$$\left|\frac{\Delta V_p}{\bar{V}_p}\right| \ll 1,$$

$$\left|\frac{\Delta V_s}{\bar{V}_s}\right| \ll 1,$$

$$\left|\frac{\Delta \rho}{\bar{\rho}}\right| \ll 1,$$

where the bar denotes an average of properties, above and below the interface. Chapman (1976) showed that the linearized reflection coefficient is (cf. also Aki and Richards, 1980):

$$R_p(\theta) = \frac{1}{2}\left[\frac{\Delta Z_p}{\bar{Z}_p}\right] + \frac{1}{2}\left[\frac{\Delta V_p}{\bar{V}_p} - \left(\frac{2\bar{V}_s}{\bar{V}_p}\right)^2 \frac{\Delta G}{\bar{G}}\right]\sin^2\theta + \frac{1}{2}\left[\frac{\Delta V_p}{\bar{V}_p}\right]\tan^2\theta\,\sin^2\theta \tag{1}$$

where θ is the incidence angle, $Z_p = \rho V_p$ is the P-wave impedance, and $G = \rho V_s^2$ is the shear modulus. Equation 1 is equivalent to, but algebraically simpler than, the linearized formulation in Shuey (1985) and differs from an expression in Wiggins et al., (1985) in not being restricted to small angles or to particular assumptions on $\bar{V}_p/\bar{V}_s$, and more importantly, in the sign of the $(\Delta V_p/\bar{V}_p)\sin^2\theta$ term.

Figure 1 shows the accuracy of the linear approximation compared to the exact solution, for a particular case. Bortfeld (1961) has presented an approximation

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for $R_p(\theta)$ which is partly linearized. In some cases, Bortfeld's approximation will be more numerically accurate than equation (1), but less accurate in other cases. In any case, numerical accuracy is not the chief virtue of equation (1); all serious calculations should be done with the exact result. The chief virtue of equation (1) is that its simplicity of form allows one to understand the physics involved. Furthermore, equation (1) shows that a numerical study (fitting data to rock properties) need only explore a three-dimensional parameter space; the three parameters are the three combinations of differentials in brackets in equation (1) or combinations of these. We are reminded, once again, that reflections do not depend on absolute values of rock properties separately, but only on certain differences in properties. Absolute values of V_p, etc., may be found by integrating (through traveltime) the differences $\Delta V_p/\bar{V}_p$, etc., found at reflecting events. Further discussion of this isotropic problem is presented elsewhere in this volume; the minus-sign in the coefficient of the $\sin^2\theta$ term in equation (1) is the crucial feature which makes this discussion interesting.

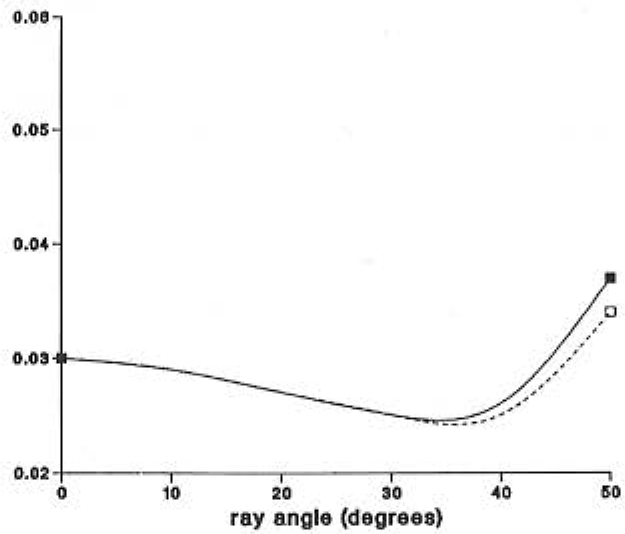

Fig. 1. Isotropic reflection coefficient variation with ray angle. Solid line—exact solution. Dashed line—weak elastic approximation [Equation (1)]. The model parameters are: incident V_p = 9500 ft/s (2895 m/s), incident V_s = 5800 ft/s (1768 m/s), incident density = 2.18 gm/cc, reflecting V_p = 10 000 ft/s (3048 m/s), reflecting V_s = 6000 ft/s (1829 m/s), reflecting density = 2.2 gm/cc.

Weak Elastic Anisotropy

Most rocks are weakly anisotropic, with elastic velocities depending both on angle of propagation (measured azimuthally and from the vertical) and on angle of polarization (for S-waves). The azimuthal dependence may be neglected, however, for P-wave propagation at small-to-moderate angles of incidence; cf. Thomsen (1988) and references cited therein. For body waves in an azimuthally isotropic rock, Thomsen (1986) showed that the various velocities normal to wavefronts are

$$V_p(\theta) = \alpha_0(1 + \delta \sin^2\theta \cos^2\theta + \varepsilon \sin^4\theta) \quad (2)$$

$$V_{S\perp}(\theta) = \beta_0\left[1 + \left(\frac{\alpha_0}{\beta_0}\right)^2 (\varepsilon - \delta) \sin^2\theta \cos^2\theta\right] \quad (3)$$

$$V_{S\parallel}(\theta) = \beta_0(1 + \gamma \sin^2\theta). \quad (4)$$

Here $S\perp$ and $S\|$ denote S-waves with polarization vectors that have a component perpendicular ($\perp$) to the plane of symmetry, or are parallel ($\|$) to the plane. If the symmetry plane is parallel to the ground surface, these waves are polarized in-line and cross-line, respectively. In a horizontally stratified isotropic context, these may be called SV- and SH-waves, respectively, but in anisotropic media, this labeling (with its irrelevant reference to the direction of gravity) may at times lead to confusion.

In equations (2, 3, and 4), α_0 and β_0 are P- and S-wave velocities, respectively, for propagation along the symmetry axis, and θ is the angle of propagation (trivially different from the wavefront-normal angle in this context). ε, δ, and γ are three independent anisotropic parameters, each assumed to be much less than one. ε is the "familiar" P-wave anisotropy; in terms of vertical and horizontal velocities

$$\varepsilon = \frac{V_p(90°) - \alpha_0}{\alpha_0}. \quad (5)$$

γ is the corresponding quantity for S-waves:

$$\gamma = \frac{V_{S\|}(90°) - \beta_0}{\beta_0}. \quad (6)$$

δ is the "strange" P-$S\perp$ anisotropic parameter, defined in terms of elastic moduli in Thomsen (1986):

$$\delta = \frac{(C_{13} + C_{44})^2 - (C_{33} - C_{44})^2}{2C_{33}(C_{33} - C_{44})}, \quad (7)$$

$$\approx \frac{C_{13} + 2C_{44} - C_{33}}{C_{33}},$$

or alternatively in terms of P-wave velocity by

$$\delta \approx 4[V_p(45°)/V_p(0°) - 1] - [V_p(90°)/V_p(0°) - 1]. \quad (8)$$

The parameter δ is intuitively inaccessible, and is tedious to measure in the laboratory, but Thomsen (1988) showed that δ is much more important in most exploration contexts than is the more familiar anisotropy parameter ε. The reason is that, for small angles θ, the ε-term in equation (2) is much smaller than the δ-term (unless in some instance $\delta \ll \varepsilon$), so that the δ-term usually dominates. Thomsen (1986) tabulated most published experimental values of δ, documenting that in general $\delta \neq \varepsilon$, and hence that P-wavefronts in most azimuthally isotropic rocks are not elliptical.

Other measures of anisotropy are possible, of course [cf. e.g., Banik (1987)]. However, the set (δ, ε γ) seems most convenient in discussions where near-vertical P-wave propagation is prominent, and hence where δ recurs as the leading anisotropy term in equations expressing P-wave angular dependence. If, in some other context, explicit introduction of a measure of $S\perp$ anisotropy were important, then the parameter

$$\sigma = \left(\frac{\alpha_0}{\beta_0}\right)^2 (\varepsilon - \delta)$$

is suggested by the form of equation (3). For near-vertical problems including $S\perp$-waves, the appropriate set of anisotropies is then (δ, σ, γ), although the redundant set (δ, σ, ε, γ) might be used if properly explained. Of course, these parameters are useful for characterizing the medium, and for intuitively understanding medium behavior, even when they are not small.

Weak Anisotropic Reflections

The solution to the general problem of P-wave reflections in anisotropic media was given in Keith and Crampin (1977). The specialization to a specific form of azimuthal anisotropy particularly relevant to seismic exploration was discussed in Thomsen (1988).

Here, we are concerned especially with angle-dependent effects due to P-wave reflections in azimuthally isotropic media, from an interface parallel to the symmetry planes of the media. The exact solution to this reflection problem was given in Daley and Hron (1977). Even more so than in the isotropic case, that solution is so algebraically complex that seeing the essential physics is difficult. Fortunately, an appropriate approximation is available, i.e., that the anisotropy is weak in both media, and that the contrasts in vertical elastic properties are also small. Specifically, the assumption is:

$$\left|\frac{\Delta\alpha_0}{\bar{\alpha}_0}\right| \ll 1 \qquad |\varepsilon_\mu| \ll 1,$$

$$\left|\frac{\Delta\beta_0}{\bar{\beta}_0}\right| \ll 1 \qquad |\delta_\mu| \ll 1, \quad (9)$$

$$\left|\frac{\Delta\rho}{\bar{\rho}}\right| \ll 1 \qquad |\gamma_\mu| \ll 1,$$

where subscript μ denotes either (1) incident medium or (2) reflecting medium.

The derivation of the quasi-P-wave reflection coefficient subject to these assumptions is given in the Appendix. The result is

$$R_p(\theta) = \frac{1}{2}\left[\frac{\Delta Z_0}{\bar{Z}_0}\right] + \frac{1}{2}\left[\frac{\Delta\alpha_0}{\bar{\alpha}_0} - \left(\frac{2\bar{\beta}_0}{\bar{\alpha}_0}\right)^2 \frac{\Delta G_0}{\bar{G}_0} + (\delta_2 - \delta_1)\right]\sin^2\theta + \frac{1}{2}\left[\frac{\Delta\alpha_0}{\bar{\alpha}_0} - (\delta_2 - \delta_1 - \varepsilon_2 + \varepsilon_1)\right]\tan^2\theta\,\sin^2\theta \quad (10)$$

where $Z_0 = \rho\alpha_0$ and $G_0 = \rho\beta_0^2$ are the vertical P-wave impedance and S-wave modulus. The corresponding transmission coefficient, and the equivalent quantities for S-waves are given in the Appendix. Equation (10) reduces to Banik's (1987) result in the limit of small angles. Figure 2 shows the accuracy of the linear approximation compared to the exact solution, for a particular case. Equation (10) is directly comparable to equation (1), the isotropic case. One sees immediately that the isotropic case may be recovered by assigning zero anisotropies in equation (10). At normal incidence ($\theta = 0$), only the first term is nonzero. This first term is the same in both equations because each anisotropic medium has been parameterized in terms of two *vertical* velocities (plus three anisotropies), instead of five elastic moduli.

The second term of both equations (in $\sin^2\theta$) gives the low-order angular variation. The coefficients of these terms involve contrasts in the vertical velocities and also contrasts in the anisotropies. The coefficient of the $\sin^2\theta$ term,

$$\frac{1}{2}\left[\frac{\Delta\alpha_0}{\bar{\alpha}_0}-\left(\frac{2\bar{\beta}_0}{\bar{\alpha}_0}\right)^2\frac{\Delta G_0}{\bar{G}_0}+(\delta_2-\delta_1)\right], \quad (11)$$

illustrates the fundamental conclusion of this work: the anisotropy is a *first-order* effect in the range dependence of $R_p(\theta)$. That is, the anisotropy enters into terms like equation (11) as simple additions (of the same magnitude) to the isotropic terms ($\Delta\alpha_0/\bar{\alpha}_0$, etc.). Therefore, the anisotropy makes a nonnegligible (first-order) contribution, even when it is weak. This happens because, while an anisotropy of 10 percent is small compared to 1, and hence negligible in many contexts, it is not negligible when the leading term is also of order 10 percent, or less, as in the present discussion, e.g., equation (11). Another example of this situation was given by Thomsen (1986) in the Discussion concluding that work.

Another feature of anisotropic reflections is that the anisotropy which appears in the lowest angular order [cf. equation (11)] is the strange anisotropy δ, rather than the familiar anisotropy ε. This is another instance of the conclusion of Thomsen (1986) that the quantity δ is more important than ε in exploration seismology. In fact, when particular examples, such as those of Wright (1984, 1987), are considered for the effects of anisotropic elastic moduli, it is primarily the combination of moduli composing δ (rather than the individual moduli) which determines the result.

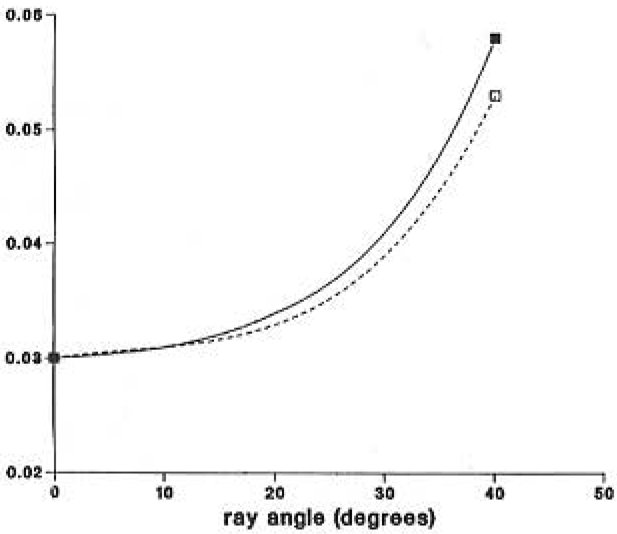

Fig. 2. Anisotropic reflection coefficient variation with ray angle. Solid line—exact solution. Dashed line—weak elastic and weak anisotropy approximation [equation (10)]. The model parameters are the same vertical velocities and densities as those of Fig. 1 and in addition incident delta (δ) = 0., incident epsilon (ε) = 0., reflecting delta (δ) = .10, and reflecting epsilon (ε) = .05.

When the exact equations (Daley and Hron, 1977) are implemented, inputting five elastic moduli for each medium, it is difficult to rationally specify modulus values to yield values of δ appropriate for sedimentary rocks. A better approach is to first select anisotropy parameters, δ, ε, and γ, perhaps using Table 1 from Thomsen (1986) as a guide. These values for intrinsic anisotropy must usually be modified by considerations of effective long-wavelength (extrinsic) anisotropy, due to stratification (Backus, 1962). Note that for δ (in contrast to ε), both positive and negative values are plausible. If the selected anisotropies δ and ε are small, one may use equation (10) directly to calculate reflectivity. A better procedure is to first calculate elastic moduli using equations (5–8), and then calculate reflectivity using the exact formulas in Daley and Hron (1977). Whether or not the anisotropies are small, equation (10) provides a guide for iterative adjustment of values for forward modeling purposes.

A third feature of the linearized anisotropic reflection coefficient, equation (10), is that the anisotropy appears only as a *difference* in anisotropies across the reflecting boundary. Hence, the isotropic case may be recovered if both media are equally anisotropic, so that differences such as $\delta_2 - \delta_1$ vanish. This is possible in some cases, but implausible in general. In fact, many reflection scenarios involve a sandstone (very weak anisotropy) on one side of the interface and a shale (weak to moderate anisotropy) on the other. In these cases, a suitable approximation is to neglect the sandstone anisotropy altogether, leaving only terms in δ_{shale} in the equations, along with the isotropic terms $\Delta\alpha_0/\bar{\alpha}_0$, etc.

Examples

The linearized equation (10) analytically verifies the conclusions of Wright (1984, 1987), determined by numerical computations using the exact solutions. In Wright's examples, one may verify that his assumed anisotropies were, in fact, plausible for sedimentary examples, by calculating

$$\varepsilon_1 = 0 \quad \delta_1 = 0$$

$$\varepsilon_2 = .26 \quad \delta_2 = .40$$

for the Model 2 shale in his Figure 5. This is a large (but not implausible) value, cf. Thomsen (1986); Wright used this value in a case where $\Delta\alpha_0/\bar{\alpha}_0$ was also large.

Wright's point may be made directly using the present analysis. The previous Figures 1 and 2, presenting isotropic and anisotropic calculations respectively, were done using the same vertical velocities (P and S) and densities in the two cases. The two calculations differ only in the non-zero anisotropies used in Figure 2. The reflecting horizon represents a normal lithologic change, i.e., one with velocity and density jumps leading to a decreasing $R_p(\theta)$, as calculated with isotropic theory (Figure 1). However, if the same normal lithologic change is associated with a plausible change in anisotropy, then $R_p(\theta)$ *increases* with angle (Figure 2) instead of decreasing. By isotropic theory, this increase could be interpreted as indicative of gas; here a similar effect is caused by anisotropy.

Banik (1987) has provided further examples. However, for developing insight it is more efficient to simply examine equation (10). One essential point is the size and sign of $\delta_2 - \delta_1$, relative to $\Delta\alpha_0/\bar{\alpha}_0$. If Wright had chosen an example with smaller $\Delta\alpha_0/\bar{\alpha}_0$, he could have shown the same anisotropic effect with a smaller δ_2. In fact, in any specific context, it is usually not clear what values of δ (and ε) are appropriate. Hence, in forward modeling, the anisotropy must often be allowed to assume a wide range of values. Therefore, in fitting the equation to data, the uncertainty in the other terms is correspondingly large, and a conclusion (such as the presence of gas in one medium) is correspondingly uncertain.

However, from equation (10), it is clear that, if the isotropic terms (e.g., $\Delta\alpha_0/\bar{\alpha}_0$) are large, then any plausible anisotropy will yield only quantitative (not qualitative) effects. Hence, among bright reflections, one may have more confidence in an isotropic interpretation than among less bright reflections, especially in a qualitative sense.

Nonetheless, *most* reflections are not bright (by definition), and so most have angular variations which are quantitatively affected by anisotropy. Hence, attempts to recover absolute values of ρ, V_s, etc., by integration over traveltime, will have to correct for these anisotropic contributions.

Implications for Exploration

It is shown here analytically that anisotropy constitutes a first-order effect in the angular dependence of P-wave reflection amplitude. Therefore, in the general case, it may not be neglected for quantitative work. In other words, the angular dependence of $R_p(\theta)$ contains information, not only about S-wave properties of the media, but also about their anisotropies. The two effects are plausibly of comparable size, and not easy to separate. Hence, the S-wave information may not be deduced from this angular dependence, without corrections or assumptions regarding the anisotropy. The assumption of zero anisotropy is probably not realistic.

The anisotropic parameter which primarily controls the effect is the "strange" anisotropy parameter δ, rather than the familiar parameter ε. Very little is known about appropriate values for δ. Thomsen (1986) tabulates most published laboratory and field measurements, but in any field context, there will be additional contributions to the effective value of δ from thin-stratification. In fact, it appears that the angular variation in $R_p(\theta)$, in conjunction with independent information on $\Delta\bar{V}_s/\bar{V}_s$, etc., may constitute one of the best ways to estimate anisotropy in situ.

Because anisotropy appears as a first-order contribution to the angular dependence of $R_p(\theta)$, it follows that exploration success, obtained by neglecting the effect in one context, may not presage similar success in other contexts. For example, the effect may be negligible (due to very small anisotropy or very similar anisotropies) in rocks in a certain basin, or of a certain age, yielding accurate predictions of gas by using the isotropic interpretation. Equation (10) shows that such success may not be confidently generalized to rocks of a different basin or a different age, where the anisotropies may not be so obliging. If the isotropic approximation is to be used, it must be justified separately in every context.

Similarly, although neglect of the anisotropic contributions may lead to qualitative success of the isotropic interpretation for bright events, this success may not extend to less bright events. Further, deduction of absolute values for in-situ S-wave velocity, etc., will be quantitatively in error, even when the isotropic theory is qualitatively adequate at selected horizons.

Therefore, it appears that the hope of obtaining cheap S-wave information from the range dependence of P-wave reflection amplitudes is complicated, in an essential way, by anisotropy. The extent of the complication should become clearer, within a few years, as the industry acquires a better understanding of the effective in-situ values of anisotropy.

APPENDIX A—DERIVATIONS

The most straightforward way of linearizing the reflection coefficient is to differentiate the exact expression for $R_p(\theta)$ [cf. Daley and Hron (1977), equation (17)] in each of the seven small quantities in equation (9) (not including γ_μ, which enters the $S\|$ problem only). Denoting these by d_n, $n = 1 \cdots 7$, the answer is

$$R_p(\theta) = \sum_n \frac{\partial R_p(\theta)}{\partial d_n} d_n \qquad \text{(A-1)}$$

where the summation is over the repeated indices. Although the algebra is tedious, it may be performed by symbolic manipulation software, such as REDUCE or Mathematica.

In practice, this straightforward approach overwhelmed the computer memory resources available at the time of this research (1982). However, substantially less memory is required by the following approach: Instead of linearizing the exact answer, we linearize the exact equations, and solve the resulting simpler problems.

The reflection coefficient $R_p(\theta)$ is found by solving the boundary conditions for continuity of displacement and stress at the interface. As in the isotropic problem, the incident P-wave excites only outgoing P- and $S\perp$-waves, with indices:

$\nu = 0$ incident quasi-P-wave
$\nu = 1$ reflected quasi-P-wave
$\nu = 2$ transmitted quasi-P-wave
$\nu = 3$ reflected quasi-$S\perp$-wave
$\nu = 4$ transmitted quasi-$S\perp$-wave

The displacement of the νth wave, propagating in the sagittal ($\hat{x} - \hat{z}$) plane, is

$$\mathbf{u}^\nu(x, z) = U^\nu \mathbf{g}_\nu e^{i(\omega t - \mathbf{k}_\nu \cdot \mathbf{x})}, \qquad \text{(A-2)}$$

where U^ν is a scalar amplitude, $\mathbf{g}_\nu$ is the νth polarization vector, ω is circular frequency, and $\mathbf{k}_\nu$ is the νth propagation vector.

The propagation vector is

$$\mathbf{k}_\nu(\theta) = \frac{\omega}{V_\nu(\theta_\nu)} [\sin\theta_\nu \hat{x} + \cos\theta_\nu \hat{z}] \qquad \text{(A-3)}$$

where V_ν is the wavefront velocity, given by equations (2) and (3) in the main text, and θ_ν is the angle between the wavefront normal and the symmetry axis. If the anisotropy is weak, the angle θ_ν differs negligibly from the ray angle. The polarization vector is

$$\mathbf{g}_\nu = (1 + \Delta\ell_\nu)\sin\theta_\nu \hat{x} + (1 + \Delta m_\nu)\cos\theta_\nu \hat{z}, \qquad \text{(A-4)}$$

where

$$\Delta\ell_\nu = [\delta_\mu - 2(\delta_\mu - \varepsilon_\mu)\sin^2\theta_\nu]\cos^2\theta_\nu, \qquad \text{(A-5)}$$

$$\Delta m_\nu = -[\delta_\mu - 2(\delta_\mu - \varepsilon_\mu)\sin^2\theta_\nu]\sin^2\theta_\nu, \qquad \text{(A-6)}$$

with index $\mu = 1$ for the incident medium ($\nu = 0, 2, 4$), and $\mu = 2$ for the reflecting medium ($\nu = 1, 3$). Using these linearized solutions of the wave equations, the boundary condition for continuity of two components of displacement at the interface ($z = 0$) may be written

$$\sum_{\nu=1}^{4} (-1)^\nu u_i^\nu = u_i^0 \quad i = 1 \text{ or } 3. \qquad \text{(A-7)}$$

The indexing scheme thus lends itself to collection of the outgoing waves on the left, and the incoming (source) wave on the right.

The corresponding equations for continuity of two components of stress, on the $z = 0$ plane, are

$$\sum_{\nu=1}^{4} (-1)^\nu \sigma_{i3}^\nu = \sigma_{i3}^0 \quad i = 1 \text{ or } 3 \qquad \text{(A-8)}$$

where the stress corresponding to the νth displacement field is

$$\sigma_{ij}^\nu(x, z) = \frac{1}{2} \sum_k \sum_\ell C_{ijk\ell}^\mu \left(\frac{\partial u_k^\nu}{\partial x_\ell} + \frac{\partial u_\ell^\nu}{\partial x_k} \right). \qquad \text{(A-9)}$$

It is easily established that no solution to equations (A-7, 8) is valid at all times, unless ω is the same for all ν, as shown. Similarly, no solution is valid at all x unless the "wavefront parameter"

$$p = \frac{\sin\theta_\nu}{V_\nu(\theta_\nu)}, \qquad \text{(A-10)}$$

is the same for all ν. This conclusion is the anisotropic generalization of Snell's law.

This set of four equations (A-7, A-8) may be written compactly following Daley and Hron (1977) (but without their interchange of rows 2 and 3) in terms of matrices:

$$\underset{\sim}{\mathbf{A}}\mathbf{U} = \mathbf{B}, \qquad \text{(A-11)}$$

where $\mathbf{U}$ is a column-vector of length four, containing the unknowns:

$$\mathbf{U} = \begin{bmatrix} U^1 \\ U^2 \\ U^3 \\ U^4 \end{bmatrix}, \qquad \text{(A-12)}$$

$\mathbf{B}$ is a column-vector containing the source terms:

$$\mathbf{B} = U_0^0 \begin{bmatrix} -pV_0(\theta_0) \\ C_{55}^1 \sin\theta_0 \cos\theta_0 \\ \cos\theta_0 \\ E_1 \end{bmatrix}. \quad \text{(A-13)}$$

The mapping matrix $\underset{\sim}{\mathbf{A}}$ has the following form [from Daley and Hron (1977)]:

$$\underset{\sim}{\mathbf{A}} = \begin{bmatrix} pV_0 & -\frac{1+\Delta\ell_2}{1+\Delta\ell_1} pV_2 & \frac{1+\Delta m_3}{1+\Delta\ell_1}\cos\theta_3 & -\frac{1+\Delta m_4}{1+\Delta\ell_1}\cos\theta_4 \\ C_{55}^1 V_0 \cos\theta_0 & C_{55}^2 LpV_0 \cos\theta_2 & C_{55}^1 W_1 & C_{55}^2 W_2 \\ \cos\theta_0 & \frac{1+\Delta m_2}{1+\Delta m_1}\cos\theta_2 & -\frac{1+\Delta\ell_3}{1+\Delta m_1} pV_3 & -\frac{1+\Delta\ell_4}{1+\Delta m_1} pV_4 \\ -E_1 & E_2 & D_1 pV_0 \cos\theta_3 & -D_2 pV_0 \cos\theta_4 \end{bmatrix}, \quad \text{(A-14)}$$

where the following symbols have been used:

$$D_1 = (1+\Delta\ell_3)C_{33}^1 - (1+\Delta m_3)C_{13}^1,$$

$$D_2 = (1+\Delta\ell_4)C_{33}^2 - (1+\Delta m_4)C_{13}^2,$$

$$E_1 = (1+\Delta\ell_1)C_{13}^1 \sin^2\theta_0 + (1+\Delta m_1)C_{33}^1 \cos^2\theta_0,$$

$$E_2 = \frac{V_1(\theta_1)}{V_2(\theta_2)}[(1+\Delta\ell_2)C_{13}^2 \sin^2\theta_2 + (1+\Delta m_2)C_{33}^2 \cos^2\theta_2],$$

$$W_1 = \frac{V_1(\theta_1)}{V_3(\theta_3)} \frac{1}{2+\Delta\ell_1+\Delta m_1} \times [(1+\Delta m_3)\cos^2\theta_3 - (1+\Delta\ell_3)\sin^2\theta_3],$$

$$W_2 = \frac{V_1(\theta_1)}{V_4(\theta_4)} \frac{1}{2+\Delta\ell_1+\Delta m_1} \times [(1+\Delta m_4)\cos^2\theta_4 - (1+\Delta\ell_4)\sin^2\theta_4],$$

$$L = \frac{2+\Delta\ell_2+\Delta m_2}{2+\Delta\ell_1+\Delta m_1}.$$

It is easy to formally write the solution to equation (A-11) as

$$\mathbf{U} = \underset{\sim}{\mathbf{A}}^{-1}\mathbf{B}. \quad \text{(A-15)}$$

Further, it is easy to rewrite the entire system in terms of the degenerate set of equations (i.e., the set with all small quantities $d_n = 0$ identically), plus small departures from this degenerate set. Denoting the degenerate set with subscript 0, the degenerate version of equation (A-11) reads

$$\underset{\sim}{\mathbf{A}}_0 \mathbf{U}_0 = \mathbf{B}_0. \quad \text{(A-16)}$$

This equation describes the propagation of a plane P-wave through a phantom "interface" between identical, isotropic media. The solution is

$$\mathbf{U}_0 = \begin{bmatrix} 0 \\ U^0 \\ 0 \\ 0 \end{bmatrix} = \underset{\sim}{\mathbf{A}}_0^{-1}\mathbf{B}_0, \quad \text{(A-17)}$$

corresponding to a transmitted P-wave with unchanged amplitude U^0.

Denoting departures from this case with Δ, the general case, equation (A-11), may be rewritten as

$$(\underset{\sim}{\mathbf{A}}_0 + \Delta\underset{\sim}{\mathbf{A}})(\mathbf{U}_0 + \Delta\mathbf{U}) = \mathbf{B}_0 + \Delta\mathbf{B}.$$

Linearizing this expression in Δ, and using equation (A-16), the difference equation is

$$\underset{\sim}{\mathbf{A}}_0 \Delta\mathbf{U} + \Delta\underset{\sim}{\mathbf{A}}\mathbf{U}_0 = \Delta\mathbf{B},$$

whose solution is

$$\Delta\mathbf{U} = \underset{\sim}{\mathbf{A}}_0^{-1}[\Delta\mathbf{B} - \Delta\underset{\sim}{\mathbf{A}}\mathbf{U}_0]. \quad \text{(A-18a)}$$

With indices explicit, this is

$$\Delta U_\nu = \sum_\kappa (A_0^{-1})_{\nu\kappa}[\Delta B_\kappa - \Delta A_{\kappa 2} U^0];$$

$$\nu = 1, 2, 3, 4. \quad \text{(A-18b)}$$

For the incident-P problem, the only nonzero element in $\mathbf{U}_0$ is the second one, [cf. equation (A-17)], which means that only the second column of differentials $\Delta A_{\kappa 2}$ is required. The first component of $\Delta\mathbf{U}$ will be $U^1(\theta)$, the scalar amplitude of the reflected P-wave; $R_p(\theta_0) = U^1(\theta)/U_0$ is the reflection coefficient.

Equation (A-18) is much easier to solve analytically than is the exact result [equation (A-15)] since $\underline{\mathbf{A}}^{-1}$ is very complex, whereas $\underline{\mathbf{A}}_0^{-1}$ may be found in simple closed form because of the high degree of symmetry in $\underline{\mathbf{A}}_0$:

$$\underline{\mathbf{A}}_0 = \begin{bmatrix} pV_0 & -A_0(1,1) & \cos\theta_3 & -A_0(1,3) \\ C^1_{55}p\alpha_0\cos\theta_1 & A_0(2,1) & \dfrac{\alpha_0 C^1_{55}(\cos^2\theta_3 - p^2\beta_0^2)}{2\beta_0} & A_0(2,3) \\ \cos\theta_1 & A_0(3,1) & -pV_3 & A_0(3,3) \\ -C^1_{33}(1-2p^2\beta_0^2) & -A_0(4,1) & 2C^1_{55}p\alpha_0\cos\theta_3 & -A_0(4,3) \end{bmatrix}, \qquad \text{(A-19)}$$

which leads to

$$\underline{\mathbf{A}}_0^{-1} = \begin{bmatrix} \left(\dfrac{\beta_0}{\alpha_0}\right)^2 \sin\theta_1 & \dfrac{p}{\rho\alpha_0\cos\theta_1} & \dfrac{1-2p^2\beta_0^2}{2\cos\theta_1} & \dfrac{-1}{2\rho\alpha_0^2} \\ -A_0^{-1}(1,1) & A_0^{-1}(1,2) & A_0^{-1}(1,3) & -A_0^{-1}(1,4) \\ \dfrac{(1-2p^2V_{s0}^2)}{2\cos\theta_3} & \dfrac{1}{\rho V_{p0}V_{s0}} & -\dfrac{V_{s0}}{V_{p0}}\sin\theta_1 & \dfrac{p}{2\rho V_{p0}\cos\theta_3} \\ -A_0^{-1}(3,1) & A_0^{-1}(3,2) & A_0^{-1}(3,3) & -A_0(3,4) \end{bmatrix} \qquad \text{(A-20)}$$

This result was found using the facilities of REDUCE (Hearn, 1978). The differentials $\Delta\mathbf{B}$ and $\Delta\underline{\mathbf{A}}$, required in equation (A-18), are straightforward, for example:

$$\Delta B_\kappa = \sum_n \frac{\partial B_\kappa}{\partial d_n} d_n\ ;\ \kappa = 1, 2, 3, 4, \qquad \text{(A-21)}$$

and similarly for $\Delta A_{\kappa 2}$. Putting these together, the answer for the reflection coefficient is given as equation (10) of the main text. Equation (10) is valid at any angle, not too close to critical, for reflections between two similar, weakly anisotropic media, with symmetry planes parallel to the interface.

The corresponding expression for the transmission coefficient is

$$T_p(\theta) = 1 - \frac{1}{2}\frac{\Delta Z_0}{\bar{Z}_0} + \frac{1}{2}\left[\frac{\Delta\alpha_0}{\bar{\alpha}_0} + (\delta_2 - \delta_1)\right]\sin^2\theta$$
$$+ \frac{1}{2}\left[\frac{\Delta\alpha_0}{\bar{\alpha}_0} + (\varepsilon_2 - \varepsilon_1)\right]\sin^2\theta\tan^2\theta$$
$$- (\delta_2 - \delta_1 - \varepsilon_2 + \varepsilon_1)\sin^4\theta \qquad \text{(A-22)}$$

Notice that $T_p \neq 1 - R_p$ except at normal incidence. Equation A-22 reduces to Banik's (1987) expression at small angles.

The corresponding expressions for $S\perp$-waves may be derived using the same methods. The results are:

$$R_{S\perp}(\theta) = -\frac{1}{2}\left(\frac{\Delta\rho}{\bar{\rho}} + \frac{\Delta\beta_0}{\bar{\beta}_0}\right) - \frac{1}{2}\frac{\Delta\beta_0}{\bar{\beta}_0}\tan^2\theta$$
$$+ 2\left(\frac{\Delta\rho}{\bar{\rho}} + 2\frac{\Delta\beta_0}{\bar{\beta}_0}\right)\sin^2\theta - \frac{1}{2}\frac{\bar{\alpha}_0^2}{\bar{\beta}_0^2}$$
$$\times (\delta_2 - \delta_1 - \varepsilon_2 + \varepsilon_1)\sin^2\theta, \qquad \text{(A-23)}$$

$$T_{S\perp}(\theta) = 1 - \frac{1}{2}\left(\frac{\Delta\rho}{\bar{\rho}} + \frac{\Delta\beta_0}{\bar{\beta}_0}\right) + \frac{1}{2}\frac{\Delta\beta_0}{\bar{\beta}_0}\tan^2\theta$$
$$- \frac{1}{2}\frac{\bar{\alpha}_0^2}{\bar{\beta}_0^2}(\delta_2 - \delta_1 - \varepsilon_2 + \varepsilon_1)$$
$$\times \sin^2\theta(1 - 2\sin^2\theta). \qquad \text{(A-24)}$$

Note that $T_{S\perp} \neq 1 - R_{S\perp}$, except at normal incidence. There is no $\sin^4$ term in $R_{S\perp}$; equation (A-23) agrees with Banik (1987). Equation (A-24) reduces to Banik's expression at small angles.

The exact coefficients for $S\|$-waves are not very complex (cf. Daley and Hron (1979). Nonetheless, it is instructive to display their form for weak anisotropy:

$$R_{S\|}(\theta) = -\frac{1}{2}\left(\frac{\Delta\rho}{\bar{\rho}} + \frac{\Delta\beta_0}{\bar{\beta}_0}\right) + \frac{1}{2}\left[\frac{\Delta\beta_0}{\bar{\beta}_0} + (\gamma_2 - \gamma_1)\right]\tan^2\theta, \quad \text{(A-25)}$$

$$T_{S\|}(\theta) = 1 + R_{S\|}(\theta). \quad \text{(A-26)}$$

Because of the ubiquity of very weak azimuthal anisotropy in sedimentary rocks, and its nonetheless marked effects on S-waves (cf. Thomsen, 1988), care should be taken to use equations (A-23 thru A-26) only in contexts where their assumptions are valid.

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